

# HOMOLOGICAL FINITENESS PROPERTIES OF WREATH PRODUCTS

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**ABSTRACT.** We study the homological finiteness properties  $FP_m$  of wreath products  $\Gamma = H \wr_X G$ . We show that, when  $H$  has infinite abelianization,  $\Gamma$  has type  $FP_m$  if and only if both  $G$  and  $H$  have type  $FP_m$  and  $G$  acts (diagonally) on  $X^i$  with stabilizers of type  $FP_{m-i}$  and with finitely many orbits for all  $1 \leq i \leq m$ .

If furthermore  $H$  is torsion-free we give a criterion for  $\Gamma$  to be Bredon- $FP_m$  with respect to the class of finite subgroups of  $\Gamma$ .

Finally, when  $H$  has infinite abelianization and  $\chi : \Gamma \rightarrow \mathbb{R}$  is a non-zero homomorphism with  $\chi(H) = 0$ , we classify when  $[\chi]$  belongs to the Bieri-Neumann-Strebel-Renz invariant  $\Sigma^m(\Gamma, \mathbb{Z})$ .

## 1. INTRODUCTION

In this paper we consider the homological finiteness property  $FP_m$  of a group. By definition a group  $G$  is of type  $FP_m$  if the trivial  $\mathbb{Z}G$ -module  $\mathbb{Z}$  has a projective resolution with all projectives finitely generated in dimensions up to  $m$ . If  $G$  is finitely presented and of type  $FP_m$  for some  $m \geq 2$  then  $G$  is of homotopical type  $F_m$  i.e. there is a  $K(G, 1)$  with finite  $m$ -skeleton.

In general it is hard to determine the homological type  $FP_m$  of a group  $G$ . Even in the case of metabelian groups  $G$  there is no complete classification of the ones of type  $FP_m$  for  $m > 2$  though there is an open conjecture, the  $FP_m$ -Conjecture, that relates the homological type  $FP_m$  with the  $m$ -element subsets of the complement of the Bieri-Strebel invariant  $\Sigma_A(Q)$  in the character sphere  $S(Q)$  [5]. Later on the Bieri-Strebel invariant was generalised for any finitely generated group  $G$  [6],[8] and is widely referred to as the Bieri-Neumann-Strebel-Renz invariant  $\Sigma^1(G, \mathbb{Z})$ . Both properties  $FP_2$  and  $F_2$  (i.e. finite presentability) coincide for a metabelian group  $G$  and the  $FP_2$ -Conjecture holds [9]. It is worth mentioning that for a general group  $G$  the properties  $FP_2$  and  $F_2$  do not coincide [2]. Though there are some sufficient conditions, see [13], there is no complete classification of finite presentability or type  $FP_2$  even in the class of nilpotent-by-abelian groups.

Let  $G, H$  be groups and  $X$  be a  $G$ -set. The (permutational) wreath product  $\Gamma = H \wr_X G$  is defined to be the semidirect product  $H^{(X)} \rtimes G$ , where  $H^{(X)}$  is the direct sum of  $|X|$  copies of  $H$  and  $G$  acts on  $H^{(X)}$  by permuting the summands. The

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classification of the finitely presented wreath products  $\Gamma = H \wr_X G$  was established in [15]. It was shown that  $\Gamma$  is finitely presented if and only if both  $H$  and  $G$  are finitely presented,  $G$  acts on  $X$  with finitely generated stabilizers, and  $G$  acts (diagonally) on  $X^2$  with finitely many orbits. We generalize this result by showing in Lemma 11 when  $G$  is of type  $FP_2$ . If  $H$  has infinite abelianization we give a criterion for  $\Gamma$  to be of type  $FP_m$  for  $m \geq 3$ . The sufficiency of the conditions of the criterion do not require that  $H$  has infinite abelianization, see Proposition 4, but our proof of the necessity of the conditions uses significantly the fact that  $H$  has infinite abelianization.

Our main results are the following theorems. The second is a homotopy version of the first one. The proof of Theorem A is homological and Theorem B is an easy corollary of Theorem A and the fact that for  $m \geq 2$  a group is of type  $F_m$  if and only if it is of type  $FP_m$  and is finitely presented.

**Theorem A** *Let  $\Gamma = H \wr_X G$  be a wreath product, where  $X \neq \emptyset$  and  $H$  has infinite abelianization. Then the following are equivalent :*

1.  $\Gamma$  is of type  $FP_m$ ;
2.  $H$  is of type  $FP_m$ ,  $G$  is of type  $FP_m$ ,  $G$  acts on  $X^i$  with stabilizers of type  $FP_{m-i}$  and with finitely many orbits for all  $1 \leq i \leq m$ .

**Theorem B** *Let  $\Gamma = H \wr_X G$  be a wreath product, where  $X \neq \emptyset$  and  $H$  has infinite abelianization. Then the following are equivalent :*

1.  $\Gamma$  is of type  $F_m$ ;
2.  $H$  is of type  $F_m$ ,  $G$  is of type  $F_m$ ,  $G$  acts on  $X^i$  with stabilizers of type  $FP_{m-i}$  and with finitely many orbits for all  $1 \leq i \leq m$ .

As a corollary we obtain some results about the Bredon- $FP_m$  type with respect to the class of finite subgroups, denoted  $\underline{FP}_m$ . This requires that  $H$  is torsion-free in order to control the conjugacy classes of finite subgroups in  $\Gamma = H \wr_X G$ . In general a group is of type  $\underline{FP}_m$  if it has finitely many conjugacy classes of finite subgroups and the centralizer of every finite subgroup is of type  $FP_m$  [17]. The homotopical counterpart of the property  $\underline{FP}_\infty$  was studied earlier in [18].

**Theorem C** *Let  $\Gamma = H \wr_X G$  be a wreath product, where  $X \neq \emptyset$  and  $H$  be torsion-free, with infinite abelianization. Then  $\Gamma$  has type  $\underline{FP}_m$  if and only if the following conditions hold :*

1.  $G$  has type  $\underline{FP}_m$ ;
2.  $H$  has type  $FP_m$ ;
3. for every finite subgroup  $K$  of  $G$  and every  $1 \leq i \leq m$  the centralizer  $C_G(K)$  acts on  $(K \backslash X)^i$  with stabilizers of type  $FP_{m-i}$  and with finitely many orbits.

*Remark.* If  $\Gamma = H \wr_X G$  has type  $\underline{FP}_m$ ,  $X \neq \emptyset$  and  $H$  has a non-trivial finite subgroup  $K$  then  $X$  is finite. Indeed since in  $\Gamma$  there are only finitely many conjugacy classes of finite subgroups there is an upper bound of the order of the finite subgroups. On other hand for every finite subset  $X_i$  of  $X$  there is a finite subgroup of  $\Gamma$  isomorphic to  $K^{X_i}$ .

Finally a monoid version of Theorem A is given in Theorem 8 for the monoid  $\Gamma_\chi = \{g \in \Gamma \mid \chi(g) \geq 0\}$ , where  $\chi : \Gamma \rightarrow \mathbb{R}$  is a non-trivial character of  $\Gamma$  such that  $\chi(H) = 0$ . This describes the points  $[\chi]$  of the Bieri-Neumann-Strebel-Renz

invariant  $\Sigma^m(\Gamma, \mathbb{Z})$  with  $\chi(H) = 0$  in terms of the invariant  $\Sigma^m(G, \mathbb{Z})$  and the action of  $G_\chi$  on  $X^i$  for  $i \leq m$ .

For a finitely generated group  $G$  the Bieri-Neumann-Strebel-Renz invariants  $\{\Sigma^m(G, \mathbb{Z})\}_{m \geq 1}$  were first defined in the case  $m = 1$  in [8] and for general  $m \geq 1$  were considered in [6]. In general the homological invariant  $\Sigma^m(G, \mathbb{Z})$  is an open subset of the unit sphere  $S(G)$  and  $\Sigma^m(G, \mathbb{Z})$  determines which subgroups of  $G$  above the commutator are of homological type  $FP_m$  [6]. The homological and homotopical  $\Sigma^m$ -invariants of a group are quite difficult to calculate but they are known for right-angled Artin groups [20], the R. Thompson group  $F$  [7], metabelian groups of finite Prüfer rank [19] and for split extensions metabelian groups if  $m = 3$  [16].

**Structure of the paper.** In Section 2 we collect some basic properties of the homological type  $FP_m$  for groups and modules. In section 3 we classify the homological type of exterior and tensor powers of induced modules. In Section 4 we classify when a group  $G = M \rtimes G$  has type  $FP_m$  provided  $M$  is a finitely generated induced  $\mathbb{Z}G$ -module. The results from Section 3 and Section 4 are applied in Section 5, where we prove Theorem A and Theorem B. In Section 7 we prove Theorem C and in Section 8 we prove a  $\Sigma$ -version of Theorem A.

## 2. PRELIMINARIES ON THE HOMOLOGICAL TYPE $FP_m$

Let  $R$  be an associative ring with unity. We recall that an  $R$ -module  $M$  is of type  $FP_m$  if  $M$  has a projective resolution with all projectives finitely generated up to dimension  $m$ . If not otherwise stated  $\otimes$  is the tensor product over  $\mathbb{Z}$  and the modules and the group actions considered are left ones.

**Lemma 1** ([3, Prop. 1.4]). Let  $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$  be a short exact sequence of  $R$ -modules.

- (a) if  $M$  and  $M_2$  are of type  $FP_s$  and  $s \geq 1$  then  $M_1$  is of type  $FP_{s-1}$ ;
- (b) if  $M_1$  and  $M_2$  are of type  $FP_s$  then  $M$  is of type  $FP_s$ ;
- (c) if  $M_1$  is of type  $FP_{s-1}$  and  $M$  is of type  $FP_s$  then  $M_2$  is of type  $FP_s$ .

The following results follow easily from Lemma 1. For completeness we include proofs.

**Lemma 2.** Let

$$\cdots \rightarrow Q_i \xrightarrow{d_i} Q_{i-1} \rightarrow \cdots \rightarrow Q_0 \xrightarrow{d_0} V \rightarrow 0$$

be an exact complex of  $R$ -modules with  $Q_i$  of type  $FP_{m-i}$  for all  $i \leq m$ . Then  $V$  has type  $FP_m$ .

*Proof.* Apply Lemma 1(c) for the short exact sequences

$$0 \rightarrow \text{im}(d_{i+1}) = \ker(d_i) \rightarrow Q_i \rightarrow \text{im}(d_i) \rightarrow 0$$

for  $0 \leq i \leq m-1$  to prove by inverse induction on  $i$  that  $\text{im}(d_i)$  is  $FP_{m-i}$ .  $\square$

**Lemma 3.** Let  $0 \leq k \leq m$  be integers and let

$$\cdots \rightarrow Q_i \xrightarrow{d_i} Q_{i-1} \rightarrow \cdots \rightarrow Q_0 \xrightarrow{d_0} V \rightarrow 0$$

be an exact complex of  $R$ -modules with  $Q_i$  of type  $FP_{m-i}$  for all  $i < k$ . Suppose further that  $V$  has type  $FP_m$ . Then  $\text{im}(d_k)$  has type  $FP_{m-k}$ .

*Proof.* Consider the exact complex

$$0 \rightarrow \text{im}(d_k) \rightarrow Q_{k-1} \xrightarrow{d_{k-1}} Q_{k-2} \rightarrow \cdots \rightarrow Q_0 \xrightarrow{d_0} V \rightarrow 0$$

and apply Lemma 1(a) for the short exact sequences associated to the above complex.  $\square$

**Lemma 4.** Let

$$\mathcal{Q}: \cdots \rightarrow Q_i \xrightarrow{d_i} Q_{i-1} \rightarrow \cdots \rightarrow Q_0 \xrightarrow{d_0} V \rightarrow 0$$

be a complex of  $R$ -modules with  $Q_i$  of type  $FP_{m-i}$  for all  $0 \leq i \leq m$  and  $H_i(\mathcal{Q})$  of type  $FP_{m-i-1}$  for  $0 \leq i \leq m-1$ . Then  $V$  has type  $FP_m$ .

*Proof.* Apply Lemma 1 for the short exact sequences

$$0 \rightarrow \ker(d_i) \rightarrow Q_i \rightarrow \text{im}(d_i) \rightarrow 0$$

and

$$0 \rightarrow \text{im}(d_{i+1}) \rightarrow \ker(d_i) \rightarrow H_i(\mathcal{Q}) \rightarrow 0$$

to prove by inverse induction on  $i$  that  $\text{im}(d_i)$  is  $FP_{m-i}$  for  $0 \leq i \leq m$ .  $\square$

The following lemma appeared as a footnote in [1] and was later on explained with more details in [10, Prop. 4.1].

**Lemma 5.** A retract of a group of type  $FP_m$  is a group of type  $FP_m$  i.e. if the split extension  $\Gamma = M \rtimes G$  is of type  $FP_m$  then  $G$  is of type  $FP_m$ .

Let  $\Gamma = M \rtimes G$  be a group. Then  $\mathbb{Z}M$  is a left  $\mathbb{Z}\Gamma$ -module, where  $G$  acts via conjugation on  $M$  and  $M$  acts on  $\mathbb{Z}M$  via left multiplication. Note that the augmentation ideal  $\text{Aug}(\mathbb{Z}M)$  is a  $\mathbb{Z}\Gamma$ -submodule of  $\mathbb{Z}M$ .

**Lemma 6.** Let  $\Gamma = M \rtimes G$  be a group. Then  $\Gamma$  is  $FP_m$  if and only if  $G$  is  $FP_m$  and  $\text{Aug}(\mathbb{Z}M)$  is  $FP_{m-1}$  as  $\mathbb{Z}\Gamma$ -module.

*Proof.* By Lemma 1 applied to the short exact sequence  $0 \rightarrow \text{Aug}(\mathbb{Z}\Gamma) \rightarrow \mathbb{Z}\Gamma \rightarrow \mathbb{Z} \rightarrow 0$  the group  $\Gamma$  is of type  $FP_m$  if and only if  $\text{Aug}(\mathbb{Z}\Gamma)$  is  $FP_{m-1}$  as  $\mathbb{Z}\Gamma$ -module. Consider the short exact sequence of  $\mathbb{Z}\Gamma$ -modules

$$(1) \quad 0 \rightarrow \mathbb{Z}\Gamma \text{Aug}(\mathbb{Z}G) \xrightarrow{\alpha} \text{Aug}(\mathbb{Z}\Gamma) \rightarrow \text{Aug}(\mathbb{Z}M) \rightarrow 0,$$

where  $\alpha$  is the inclusion map.

If  $\Gamma$  is of type  $FP_m$  by Lemma 5  $G$  is of type  $FP_m$ , so  $\text{Aug}(\mathbb{Z}G)$  is  $FP_{m-1}$  as  $\mathbb{Z}G$ -module and the induced  $\mathbb{Z}\Gamma$ -module  $\mathbb{Z}\Gamma \text{Aug}(\mathbb{Z}G)$  is  $FP_{m-1}$ . Then by Lemma 1(c) applied to the short exact sequence (1)  $\text{Aug}(\mathbb{Z}M)$  is  $FP_{m-1}$  as  $\mathbb{Z}\Gamma$ -module.

If  $G$  is of type  $FP_m$  and  $\text{Aug}(\mathbb{Z}M)$  is  $FP_{m-1}$  as  $\mathbb{Z}\Gamma$ -module then by Lemma 1(b) applied to the short exact sequence (1)  $\text{Aug}(\mathbb{Z}\Gamma)$  is  $FP_{m-1}$  as  $\mathbb{Z}\Gamma$ -module.  $\square$

**Lemma 7.** Let  $S$  be a subring of  $R$  such that  $R$  is flat as right  $S$ -module and  $M$  be a left  $S$ -module. Furthermore assume that the inclusion map  $S \rightarrow R$  of right  $S$ -modules splits. Then  $M$  is of type  $FP_m$  as  $S$ -module if and only if  $R \otimes_S M$  is  $FP_m$  as  $R$ -module.

*Proof.* 1. Suppose that  $M$  is of type  $FP_m$  as  $S$ -module. Let  $\mathcal{F}$  be a projective resolution of  $M$  with projectives finitely generated in dimensions  $\leq m$ . Since  $R \otimes_S -$  is an exact functor  $R \otimes_S \mathcal{F}$  is a projective resolution of the  $R$ -module  $R \otimes_S M$  with projectives finitely generated in dimensions  $\leq m$ , so  $R \otimes_S M$  is  $FP_m$  as  $R$ -module.

2. Suppose that  $R \otimes_S M$  is  $FP_m$  as  $R$ -module. We will prove by induction on  $m$  that  $M$  is  $FP_m$  as  $S$ -module.

Suppose first that  $m = 0$ . Let  $X = \{a_i = \sum_j r_{i,j} \otimes m_{i,j}\}_i$  be a finite generating set of  $R \otimes_S M$  as  $R$ -module, where  $r_{i,j} \in R, m_{i,j} \in M$ . Let  $M_0$  be the  $S$ -submodule of  $M$  generated by the finite set  $Y = \{m_{i,j}\}_{i,j}$ . Since  $R$  is flat as  $S$ -module there is a short exact sequence  $0 \rightarrow R \otimes_S M_0 \xrightarrow{\alpha} R \otimes_S M \rightarrow R \otimes_S (M/M_0) \rightarrow 0$  and by the choice of  $Y$  the map  $\alpha$  is surjective, so  $\alpha$  is an isomorphism. Then  $R \otimes_S (M/M_0) = 0$  has a direct summand  $S \otimes_S (M/M_0) = M/M_0$ , so  $M_0 = M$ .

Suppose now that  $M$  is  $FP_{m-1}$  as  $S$ -module. Let

$$\mathcal{F} : \cdots \rightarrow F_i \xrightarrow{d_i} F_{i-1} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0$$

be a projective resolution of  $S$ -modules with  $F_i$  finitely generated for  $i \leq m-1$ . Consider the exact complex induced by  $\mathcal{F}$

$$\mathcal{R} : 0 \rightarrow A = \ker(d_{m-1}) \rightarrow F_{m-1} \xrightarrow{d_{m-1}} F_{m-2} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0.$$

Since  $R$  is flat as right  $S$ -module

$$R \otimes_S \mathcal{R} : 0 \rightarrow R \otimes_S A \rightarrow R \otimes_S F_{m-1} \xrightarrow{id_R \otimes d_{m-1}} R \otimes_S F_{m-2} \rightarrow \cdots \rightarrow R \otimes_S F_0 \rightarrow R \otimes_S M \rightarrow 0$$

is an exact complex and  $R \otimes_S F_i$  is a finitely generated projective  $R$ -module for  $i \leq m-1$ . By Lemma 3 since  $R \otimes_S M$  is  $FP_m$  as  $R$ -module we have that  $R \otimes_S A$  is finitely generated as  $R$ -module. Then by the case  $m = 0$  we deduce that  $A$  is finitely generated as  $S$ -module, so  $M$  is  $FP_m$  as  $S$ -module.  $\square$

**Corollary 1.** Let  $L$  be a subgroup of a group  $G$  and  $A$  be a (left)  $\mathbb{Z}L$ -module. Then  $\mathbb{Z}G \otimes_{\mathbb{Z}L} A$  is  $FP_m$  as  $\mathbb{Z}G$ -module if and only if  $A$  is  $FP_m$  as  $\mathbb{Z}L$ -module.

### 3. EXTERIOR AND TENSOR POWERS OF INDUCED MODULES

Let  $G$  be a group and  $S$  be a free  $\mathbb{Z}$ -module with basis  $X \neq \emptyset$ . If  $S$  is a  $\mathbb{Z}G$ -module via an action of  $G$  on the set  $X$ , we call  $S$  an induced module. If  $S$  is a  $\mathbb{Z}G$ -module via an action of  $G$  on the set  $X \cup -X$  we say that  $S$  is a semi-induced module, in this case we write  $S = \mathbb{Z}_t X$  with index  $t$  standing for twisted action of  $G$ .

**Lemma 8.** Let  $S = \mathbb{Z}_t X$  be a finitely generated semi-induced  $\mathbb{Z}G$ -module. Then

(a)  $S \simeq \bigoplus_{i \in I} \mathbb{Z}G \otimes_{\mathbb{Z}K_i} V_i$  as  $\mathbb{Z}G$ -modules where  $I$  is finite, for each  $i \in I$  there is an element  $x_i \in X$  with stabilizer  $G_i$  in  $G$  such that  $K_i$  is a subgroup of  $G$  containing  $G_i$  such that  $[K_i : G_i] = d_i \in \{1, 2\}$  and each  $V_i$  as an abelian group is  $\mathbb{Z}$ ,  $G_i$  acts trivially on  $V_i$  and  $K_i$  acts non-trivially on  $V_i$  precisely when  $d_i = 2$ .

(b)  $S$  is of type  $FP_m$  over  $\mathbb{Z}G$  if and only if all the groups  $G_i$  are of type  $FP_m$  for  $i \in I$ .

*Proof.* (a) Observe first that  $S$  splits as a finite direct sum of cyclic semi-induced  $\mathbb{Z}G$ -modules, each generated by an element of  $X$ . Thus from the very beginning we may assume that  $S$  is a cyclic semi-induced  $\mathbb{Z}G$ -module with a generator  $w$  an element from  $X$ . If  $S$  is an induced  $\mathbb{Z}G$ -module we are done, so assume that  $S$  is not an induced  $\mathbb{Z}G$ -module. Then there are  $g_1, g_2$  in  $G$  such that  $g_1 w = -g_2 w$ , so for  $t = g_2^{-1} g_1$  we have  $tw = -w$ . Then  $t^2 w = t(-w) = -tw = -(-w) = w$ , so  $t^2 \in G_0$ , where  $G_0$  is the stabilizer of  $w$  in  $G$ . For  $g_0 \in G_0$  we have  $g_0 t w = -g_0 w = -w = tw$ , so  $t^{-1} g_0 t \in G_0$ . Then for the subgroup  $K_0$  of  $G$  generated by  $G_0$  and  $t$  we have that  $G_0$  is a normal subgroup of  $K_0$  of index 2.

Observe that if  $h_1, h_2 \in G$  and  $h_1 w = -h_2 w$  then for  $t_1 = h_2^{-1} h_1$  we have  $t_1 w = -w = tw$ , so  $t_1^{-1} t \in G_0$ . Thus  $h_1 w = -h_2 w$  is a corollary of  $tw = -w$  so  $S \simeq (\mathbb{Z}G \otimes_{\mathbb{Z}G_0} \mathbb{Z}) / \mathbb{Z}G(tG_0 + G_0) \simeq \mathbb{Z}G \otimes_{\mathbb{Z}K_0} V_0$ , where  $V_0 \simeq (\mathbb{Z}K_0 \otimes_{\mathbb{Z}G_0} \mathbb{Z}) / \mathbb{Z}K_0(G_0 + tG_0)$  has underlying abelian group  $\mathbb{Z}$  and  $G_0$  acts trivially and  $K_0/G_0$  changes the sign. This completes the proof of item (a).

(b) Observe that since  $S \simeq \bigoplus_{i \in I} \mathbb{Z}G \otimes_{\mathbb{Z}K_i} V_i$  we have that  $S$  is of type  $FP_m$  if and only if  $\mathbb{Z}G \otimes_{\mathbb{Z}K_i} V_i$  is  $FP_m$  as  $\mathbb{Z}G$ -module for every  $i \in I$ . By Corollary 1  $\mathbb{Z}G \otimes_{\mathbb{Z}K_i} V_i$  is  $FP_m$  as  $\mathbb{Z}G$ -module precisely when  $V_i$  is of type  $FP_m$  over  $\mathbb{Z}K_i$ . If  $K_i = G_i$  we have that  $V_i$  is the trivial  $\mathbb{Z}G_i$ -module, so  $V_i$  is of type  $FP_m$  over  $\mathbb{Z}K_i$  precisely when  $G_i = K_i$  is  $FP_m$  as a group. If  $K_i/G_i$  is the cyclic group of order 2 then  $V_i$  is the sign module i.e.  $G_i$  acts trivially on  $V_i = \mathbb{Z}$  and  $K_i/G_i$  acts via sign change. Since  $G_i$  has finite index in  $K_i$  we have that  $V_i$  is  $FP_m$  as  $\mathbb{Z}K_i$ -module precisely when  $V_i$  is  $FP_m$  as  $\mathbb{Z}G_i$ -module, but as  $\mathbb{Z}G_i$ -module  $V_i$  is the trivial one, so  $V_i$  is  $FP_m$  as  $\mathbb{Z}G_i$ -module is the same as  $G_i$  is  $FP_m$ .  $\square$

The following proposition is the main result of this section. It classifies the homological type of tensor and exterior powers of induced modules.

**Proposition 1.** Let  $M = \mathbb{Z}X$  be an induced  $\mathbb{Z}G$ -module. Then the following are equivalent :

1.  $\bigwedge^i M$  is of type  $FP_{m-i}$  for all  $1 \leq i \leq m$ ;
2.  $\bigotimes^i M$  is of type  $FP_{m-i}$  for all  $1 \leq i \leq m$ ;
3. all stabilizers of the diagonal action of  $G$  on  $X^i$  are of type  $FP_{m-i}$  and  $\bigotimes^i M$  is finitely generated as  $\mathbb{Z}G$ -module for all  $1 \leq i \leq m$ ;
4. all stabilizers of  $i$  element subsets of  $X$  are of type  $FP_{m-i}$  and  $\bigwedge^i M$  is finitely generated as  $\mathbb{Z}G$ -module for all  $1 \leq i \leq m$ ;
5. all stabilizers of the diagonal action of  $G$  on  $X^i$  are of type  $FP_{m-i}$  and  $G \backslash X^i$  is finite for all  $1 \leq i \leq m$ .

*Proof.* Observe that  $\bigwedge^i M = \mathbb{Z}_i Y_i$  is an semi-induced  $\mathbb{Z}G$ -module, where  $Y_i$  is the set of  $i$ -element subsets of  $X$  and  $\bigotimes^i M = \mathbb{Z}Z_i$  is an induced  $\mathbb{Z}G$ -module, where  $Z_i = X^i$ .

It is obvious that conditions 2,3 and 5 are equivalent.

Note that the stabilizer of  $(x_1, \dots, x_i) \in X^i$  in  $G$  (via the diagonal action) is commensurable with the stabilizer of the set  $\{x_1, \dots, x_i\}$  in  $G$ , in particular one of the stabilizers is  $FP_{m-i}$  precisely when the other is  $FP_{m-i}$ . Furthermore  $\bigwedge^i M$  is finitely generated as  $\mathbb{Z}G$ -module for all  $i \leq m$  if and only if  $G$  acts on the set of  $i$  element subsets of  $X$  with finitely many orbits for all  $i \leq m$ . The last is equivalent with  $G$  acts on  $X^i$  with finitely many orbits for all  $i \leq m$ . Thus conditions 5 and 4 are equivalent, and by Lemma 8(b) conditions 1 and 2 are equivalent too.  $\square$

**Lemma 9.** Let  $X$  be a (left)  $G$ -set such that  $G$  acts with finitely many orbits on  $X^i$  and with stabilizers of type  $FP_{m-i}$  for all  $1 \leq i \leq m$ . Let  $i_1, i_2, \dots, i_s$  be non-negative integers such that  $i_1 + \dots + i_s = r \leq m$  and  $X^{(i_1, i_2, \dots, i_s)}$  be the set of  $s$ -tuples  $(X_1, \dots, X_s)$  of pairwise disjoint subsets  $X_1, \dots, X_s$  of  $X$  such that  $X_j$  has cardinality  $i_j$  for all  $1 \leq j \leq s$ . Then  $\mathbb{Z}X^{(i_1, i_2, \dots, i_s)}$  is of type  $FP_{m-r}$  as  $\mathbb{Z}G$ -module.

*Proof.* Let  $B$  be the stabilizer in  $G$  of the  $s$ -tuple  $(X_1, \dots, X_s)$  i.e.  $B$  is the set of  $g \in G$  such that  $gX_i = X_i$  for all  $1 \leq i \leq s$ . Then since all  $X_i$  are finite  $B$  is a subgroup of finite index in the group  $D$ , where  $D$  is the stabilizer in  $G$  of the set  $\bigcup_{1 \leq i \leq s} X_i$ . Note that  $|\bigcup_{1 \leq i \leq s} X_i| = i_1 + \dots + i_s = r$ . Then by Proposition 1 (i.e.

item 5 implies item 4),  $D$  is of type  $FP_{m-r}$ , hence  $B$  is of type  $FP_{m-r}$ . Note that  $G \setminus X^r$  finite implies that  $G \setminus X^{(i_1, i_2, \dots, i_s)}$  is finite. Then by Lemma 8 the  $\mathbb{Z}G$ -module  $\mathbb{Z}X^{(i_1, i_2, \dots, i_s)}$  is of type  $FP_{m-r}$ .  $\square$

#### 4. TYPE $FP_m$ FOR SPLIT EXTENSIONS OF INDUCED MODULES

Let  $\Gamma = M \rtimes G$  be a group, where  $M$  is abelian. Note we do not impose restrictions on  $G$ . We consider  $M$  as a left  $\mathbb{Z}G$ -module, where  $G$  acts by conjugation. In this section  $M$  is an induced finitely generated (left)  $\mathbb{Z}G$ -module i.e.

$$(2) \quad M = \bigoplus_{i \in I} \mathbb{Z}G \otimes_{\mathbb{Z}H_i} \mathbb{Z} \simeq \bigoplus_{i \in I} \mathbb{Z}[G/H_i] = \mathbb{Z}X,$$

where  $X = \bigcup_{i \in I} G/H_i$ ,  $I$  is a finite set and  $\{H_i\}_{i \in I}$  are the stabilizers of the action of  $G$  on  $X$ . In this section we classify when a group  $\Gamma = M \rtimes G$  is of type  $FP_m$  provided  $M$  is an induced finitely generated  $\mathbb{Z}G$ -module.

**Proposition 2.** Let  $G$  be a group of type  $FP_m$  and  $M$  be a finitely generated induced  $\mathbb{Z}[G]$ -module such that  $\wedge^i M$  be of type  $FP_{m-i}$  as  $\mathbb{Z}G$ -module via the diagonal  $G$ -action for all  $1 \leq i \leq m$ . Then  $\Gamma = M \rtimes G$  is of type  $FP_m$ .

*Proof.* Let  $X$  be the disjoint union  $\bigcup_i G/H_i$ , so  $X$  is a basis of  $M$  as a free  $\mathbb{Z}$ -module. Since  $M$  is a torsion-free abelian group, there is a Koszul complex [22]

$$(3) \quad \cdots \rightarrow \mathbb{Z}M \otimes_{\mathbb{Z}} (\wedge^k M) \xrightarrow{d_k} \mathbb{Z}M \otimes_{\mathbb{Z}} (\wedge^{k-1} M) \rightarrow \cdots \rightarrow \mathbb{Z}M \otimes_{\mathbb{Z}} M \rightarrow \mathbb{Z}M \rightarrow \mathbb{Z} \rightarrow 0$$

with differential given by

$$d_k(m_1 \wedge \cdots \wedge m_k) = \sum_{1 \leq i \leq k} (-1)^i \epsilon(m_i) \otimes m_1 \wedge \cdots \wedge m_{i-1} \wedge m_{i+1} \wedge \cdots \wedge m_k,$$

where  $\epsilon(m_i) = m_i - 1 \in \mathbb{Z}M$  and  $m_1, \dots, m_k \in X$ . Thus (3) gives an exact complex

$$(4) \quad \begin{aligned} \mathcal{Q} : \cdots \rightarrow Q_{k-1} = \mathbb{Z}M \otimes_{\mathbb{Z}} (\wedge^k M) \xrightarrow{d_k} Q_{k-2} = \mathbb{Z}M \otimes_{\mathbb{Z}} (\wedge^{k-1} M) \rightarrow \cdots \\ \rightarrow Q_0 = \mathbb{Z}M \otimes_{\mathbb{Z}} M \rightarrow Q_{-1} = \text{Aug}(\mathbb{Z}M) \rightarrow 0. \end{aligned}$$

Note that for a left  $\mathbb{Z}G$ -module  $V$  there is an isomorphism of abelian groups

$$\mathbb{Z}\Gamma \otimes_{\mathbb{Z}G} V \simeq \mathbb{Z}M \otimes_{\mathbb{Z}} V$$

and thus the action of  $\Gamma$  on the induced module  $\mathbb{Z}\Gamma \otimes_{\mathbb{Z}G} V$  gives an action of  $\Gamma$  on  $\mathbb{Z}M \otimes_{\mathbb{Z}} V$ . We apply this for  $V = \wedge^k M$  and since  $\wedge^k M$  is of type  $FP_{m-k}$  as  $\mathbb{Z}G$ -module and  $\mathbb{Z}\Gamma \otimes_{\mathbb{Z}G}$  is an exact functor

$$\mathbb{Z}\Gamma \otimes_{\mathbb{Z}G} (\wedge^k M) \simeq \mathbb{Z}M \otimes_{\mathbb{Z}} (\wedge^k M)$$

is of type  $FP_{m-k}$  as  $\mathbb{Z}\Gamma$ -module. Note that since  $X$  is a  $G$ -invariant set the differentials in (4) are homomorphisms of  $\mathbb{Z}G$ -modules, hence are homomorphisms of  $\mathbb{Z}\Gamma$ -modules. Then (4) is an exact complex with  $Q_i$  of homological type  $FP_{m-i-1}$  as  $\mathbb{Z}\Gamma$ -module for  $0 \leq i \leq m-1$ . Then by Lemma 2 applied to the complex (4)  $\text{Aug}(\mathbb{Z}M)$  is of type  $FP_{m-1}$  as  $\mathbb{Z}\Gamma$ -module, hence by Lemma 6  $\Gamma$  is of type  $FP_m$ .  $\square$

**Lemma 10.** Suppose that  $\Gamma = M \rtimes G$  is a group of type  $FP_m$  and  $M$  is a finitely generated induced  $\mathbb{Z}[G]$ -module such that  $\wedge^i M$  is of type  $FP_{m-i-1}$  as  $\mathbb{Z}G$ -module (via the diagonal  $G$ -action) for all  $1 \leq i \leq m-1$ . Then  $M$  is of type  $FP_{m-1}$  as  $\mathbb{Z}G$ -module.

*Proof.* By Lemma 6  $\Omega = \text{Aug}(\mathbb{Z}M)$  is of type  $FP_{m-1}$  as  $\mathbb{Z}\Gamma$ -module. Note that  $M \simeq \Omega/\Omega^2 \simeq \mathbb{Z} \otimes_{\mathbb{Z}M} \text{Aug}(\mathbb{Z}M)$ . Let

$$\mathcal{F} : \cdots \rightarrow F_i \rightarrow F_{i-1} \rightarrow \cdots \rightarrow F_0 \rightarrow \Omega \rightarrow 0$$

be a free resolution of  $\Omega$  as  $\mathbb{Z}\Gamma$ -module with  $F_i$  finitely generated for  $i \leq m-1$ . Let  $\mathcal{Q}$  be the complex  $\mathbb{Z} \otimes_{\mathbb{Z}M} \mathcal{F}$  i.e.

$$(5) \quad \mathcal{Q} : \cdots \rightarrow Q_i \rightarrow Q_{i-1} \rightarrow \cdots \rightarrow Q_0 \rightarrow M \rightarrow 0$$

is a complex of free  $\mathbb{Z}G$ -modules with  $Q_i$  finitely generated for  $i \leq m-1$  and its homology groups are

$$H_i(\mathcal{Q}) \simeq \text{Tor}_i^{\mathbb{Z}M}(\mathbb{Z}, \Omega) \text{ for } i \geq 1.$$

Note that the long exact sequence in Tor applied to the short exact sequence  $\Omega \rightarrow \mathbb{Z}M \rightarrow \mathbb{Z}$  gives

$$\text{Tor}_i^{\mathbb{Z}M}(\mathbb{Z}, \Omega) \simeq \text{Tor}_{i+1}^{\mathbb{Z}M}(\mathbb{Z}, \mathbb{Z}) = H_{i+1}(M, \mathbb{Z}) \simeq \wedge^{i+1} M \text{ for } i \geq 1.$$

The last isomorphism is [11, Ch. V, Thm. 6.4]. Then  $H_i(\mathcal{Q})$  is of type  $FP_{m-i-2}$  as  $\mathbb{Z}G$ -module for all  $i \leq m-2$ . Then by Lemma 4 applied to the complex (5)  $M$  is of type  $FP_{m-1}$  as  $\mathbb{Z}G$ -module.  $\square$

**Proposition 3.** Let  $\Gamma = M \rtimes G$  be a group of type  $FP_m$  and  $M$  be a finitely generated induced  $\mathbb{Z}[G]$ -module. Then  $\wedge^i M$  is of type  $FP_{m-i}$  as  $\mathbb{Z}G$ -module via the diagonal  $G$ -action for all  $1 \leq i \leq m$ .

*Proof.* We induct on  $m$  and assume that the proposition holds for smaller values of  $m$ , in particular  $\wedge^i M$  is  $FP_{m-i-1}$  as  $\mathbb{Z}G$ -module. By Lemma 10  $M$  is of type  $FP_{m-1}$  as  $\mathbb{Z}G$ -module, thus the proposition holds for  $i = 1$ . We proceed by induction on  $i$  i.e. assume that we have proved that  $\wedge^j M$  is of type  $FP_{m-j}$  as  $\mathbb{Z}G$ -module via the diagonal  $G$ -action for all  $1 \leq j \leq i-1$  and will show that  $\wedge^i M$  is of type  $FP_{m-i}$  as  $\mathbb{Z}G$ -module.

Consider the Koszul complex

$$(6) \quad \cdots \rightarrow \mathbb{Z}M \otimes_{\mathbb{Z}} (\wedge^k M) \xrightarrow{d_k} \mathbb{Z}M \otimes_{\mathbb{Z}} (\wedge^{k-1} M) \rightarrow \cdots \rightarrow \mathbb{Z}M \otimes_{\mathbb{Z}} M \xrightarrow{d_1} \mathbb{Z}M \rightarrow \mathbb{Z} \rightarrow 0$$

and its modified version

$$(7) \quad \begin{aligned} \mathcal{Q} : \cdots \rightarrow Q_{k-1} &= \mathbb{Z}M \otimes_{\mathbb{Z}} (\wedge^k M) \xrightarrow{d_k} Q_{k-2} = \mathbb{Z}M \otimes_{\mathbb{Z}} (\wedge^{k-1} M) \rightarrow \cdots \\ &\rightarrow Q_0 = \mathbb{Z}M \otimes_{\mathbb{Z}} M \xrightarrow{d_1} Q_{-1} = \text{Aug}(\mathbb{Z}M) \rightarrow 0 \end{aligned}$$

Since  $\wedge^k M$  is  $FP_{m-k}$  as  $\mathbb{Z}G$ -module for  $k \leq i-1$  the induced module  $Q_{k-1} = \mathbb{Z}M \otimes_{\mathbb{Z}} (\wedge^k M)$  is  $FP_{m-k}$  as  $\mathbb{Z}\Gamma$ -module for  $k \leq i-1$ . Note that by Lemma 6  $\text{Aug}(\mathbb{Z}M)$  is  $FP_{m-1}$  as  $\mathbb{Z}\Gamma$ -module. Then by Lemma 3 applied to the complex (4)  $\text{im}(d_i)$  is of type  $FP_{m-1-(i-1)}$  as  $\mathbb{Z}\Gamma$ -module. Note that

$$\text{im}(d_i) \simeq (\mathbb{Z}M \otimes (\wedge^i M)) / \ker(d_i) \simeq (\mathbb{Z}M \otimes (\wedge^i M)) / \text{im}(d_{i+1}).$$

Denote  $(\mathbb{Z}M \otimes (\wedge^i M)) / \text{im}(d_{i+1})$  by  $V$  i.e.  $V$  has type  $FP_{m-i}$  as  $\mathbb{Z}\Gamma$ -module. Let  $W$  be  $\mathbb{Z} \otimes_{\mathbb{Z}M} V$ . Since  $\mathbb{Z} \otimes_{\mathbb{Z}M} -$  is right exact there is an exact sequence

$$\mathbb{Z} \otimes_{\mathbb{Z}M} \text{im}(d_{i+1}) \xrightarrow{\beta} \mathbb{Z} \otimes_{\mathbb{Z}M} (\mathbb{Z}M \otimes (\wedge^i M)) \rightarrow W \rightarrow 0.$$

By the definition of  $d_{i+1}$  we have  $\beta = 0$ , furthermore the module in the middle is isomorphic to  $\wedge^i M$ , so

$$W \simeq \wedge^i M.$$

If  $W$  is of type  $FP_{m-i}$  as  $\mathbb{Z}G$ -module the inductive step is completed.



Let

$$\mathcal{F} : \cdots \rightarrow F_j \rightarrow F_{j-1} \rightarrow \cdots \rightarrow F_0 \rightarrow V \rightarrow 0$$

be a free resolution of  $V$  as  $\mathbb{Z}\Gamma$ -module with  $F_j$  finitely generated for  $j \leq m-i$ . Let  $\mathcal{R}$  be the complex  $\mathbb{Z} \otimes_{\mathbb{Z}M} \mathcal{F}$  i.e.

$$\mathcal{R} : \cdots \rightarrow R_j \rightarrow R_{j-1} \rightarrow \cdots \rightarrow R_0 \rightarrow W \rightarrow 0$$

is a complex of free  $\mathbb{Z}G$ -modules with  $R_j$  finitely generated for  $j \leq m-i$  and its homology groups are

$$H_j(\mathcal{R}) \simeq \text{Tor}_j^{\mathbb{Z}M}(\mathbb{Z}, V) \text{ for } j \geq 1.$$

Note that for the  $\mathbb{Z}M$ -module  $V = \text{im}(d_i)$  the Koszul complex (6) gives a free resolution

$$(8) \quad \begin{aligned} \mathcal{C} : \cdots \rightarrow \mathbb{Z}M \otimes_{\mathbb{Z}} (\wedge^k M) \xrightarrow{d_k} \mathbb{Z}M \otimes_{\mathbb{Z}} (\wedge^{k-1} M) \rightarrow \cdots \rightarrow \\ \mathbb{Z}M \otimes_{\mathbb{Z}} (\wedge^{i+1} M) \rightarrow \mathbb{Z}M \otimes_{\mathbb{Z}} (\wedge^i M) \xrightarrow{d_i} V \rightarrow 0 \end{aligned}$$

and by the definition of the differentials  $d_k$  the complex

$$(9) \quad \mathbb{Z} \otimes_{\mathbb{Z}M} \mathcal{C} : \cdots \rightarrow \wedge^{i+2} M \rightarrow \wedge^{i+1} M \rightarrow \wedge^i M \xrightarrow{\alpha} \mathbb{Z} \otimes_{\mathbb{Z}M} \text{im}(d_i) = W \rightarrow 0$$

has all zero differentials except possibly  $\alpha$ . Thus

$$H_j(\mathcal{R}) \simeq \text{Tor}_j^{\mathbb{Z}M}(\mathbb{Z}, V) \simeq H_j(\mathbb{Z} \otimes_{\mathbb{Z}M} \mathcal{C}) \simeq \wedge^{j+i} M \text{ for } j \geq 1.$$

By Lemma 4 applied to the complex  $\mathcal{R}$ , for  $W$  to be of type  $FP_{m-i}$  as  $\mathbb{Z}G$ -module it is sufficient that

$$H_j(\mathcal{R}) \text{ is of type } FP_{m-i-j-1} \text{ as } \mathbb{Z}G\text{-module for } 1 \leq j \leq m-i-1.$$

Thus we need for  $k = i+j$  that

$$\wedge^k M \text{ is of type } FP_{m-k-1} \text{ as } \mathbb{Z}G\text{-module for } 1 \leq k \leq m-1$$

but this follows from the fact that the proposition holds for  $m-1$ .  $\square$

The following theorem is the main result of this section and classifies when a group  $\Gamma = M \rtimes G$  is of type  $FP_m$  provided  $M$  is an induced finitely generated  $\mathbb{Z}G$ -module.

**Theorem 1.** Let  $\Gamma = M \rtimes G$  be a group and  $M$  be a finitely generated induced  $\mathbb{Z}[G]$ -module. Then  $\Gamma$  is of type  $FP_m$  if and only if  $G$  is of type  $FP_m$  and  $\wedge^i M$  is of type  $FP_{m-i}$  as  $\mathbb{Z}G$ -module via the diagonal action for all  $1 \leq i \leq m$ .

*Proof.* Follows directly from Lemma 5, Proposition 2 and Proposition 3.  $\square$

## 5. HOMOLOGICAL TYPE $FP_m$ FOR WREATH PRODUCTS

**5.1. Preliminaries on tensor products of complexes.** Let  $(\mathcal{A}, \partial)$  be a non-negative complex of free  $\mathbb{Z}$ -modules i.e. all non-zero modules are in dimension  $\geq 0$ . Let  $\mathcal{A}_i = \mathcal{A}$  for all  $1 \leq i \leq s$ . Consider the tensor product  $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_s$  with differential given by

$$(10) \quad d(a_1 \otimes \cdots \otimes a_s) = \sum_{1 \leq j \leq s} (-1)^{\deg(a_1) + \cdots + \deg(a_j)} a_1 \otimes \cdots \otimes \partial(a_j) \otimes \cdots \otimes a_s.$$

For the transposition  $\pi = (i, i+1) \in S_s$  and each  $a_i$  an element of one of the free modules in  $\mathcal{A}_i$  we define

$$(11) \quad \pi(a_1 \otimes \cdots \otimes a_s) := (-1)^{\deg(a_i) \deg(a_{i+1})} (a_1 \otimes \cdots \otimes a_{i+1} \otimes a_i \otimes \cdots \otimes a_s).$$

Since the symmetric group  $S_s$  is generated by  $\{(i, i+1)\}_{1 \leq i \leq s-1}$  we get an action of  $S_s$  on the tensor product  $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_s$ . As the action of the symmetric group  $S_s$  commutes with the differential from (10), this induces an action of  $S_s$  on the homology groups of  $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_s$ .

**5.2. Wreath products : proofs of Theorem A and Theorem B.** In this section we classify when a wreath product  $\Gamma = H \wr_X G$  is of type  $FP_m$ . First we show in Proposition 4 some sufficient conditions for  $\Gamma$  to be of type  $FP_m$ . The difficult part of the classification is to show that these sufficient conditions are necessary. We establish this in Theorem 2 under the extra condition that  $H$  has infinite abelianization.

**Proposition 4.** Let  $\Gamma = H \wr_X G$  be a wreath product such that both  $H$  and  $G$  are of type  $FP_m$ ,  $G$  acts (diagonally) on  $X^i$  with stabilizers of type  $FP_{m-i}$  and with finitely many orbits for  $1 \leq i \leq m$ . Then  $\Gamma$  has type  $FP_m$ .

*Proof.* Let

$$\mathcal{H} : \cdots \rightarrow \mathbb{Z}H \otimes \mathbb{Z}Y_i \xrightarrow{\partial_i} \mathbb{Z}H \otimes \mathbb{Z}Y_{i-1} \rightarrow \cdots \rightarrow \mathbb{Z}H \otimes \mathbb{Z}Y_1 \xrightarrow{\partial_1} \mathbb{Z}H \rightarrow \mathbb{Z} \rightarrow 0$$

be a free resolution of the trivial  $\mathbb{Z}H$ -module  $\mathbb{Z}$  with all  $Y_i$  finite for  $i \leq m$ .

Consider  $H_x$  an isomorphic copy of the group  $H$  and let  $\mathcal{H}_x$  be the complex obtained from  $\mathcal{H}$  by substituting  $H$  with  $H_x$ . We write  $\partial_x$  for the differential of  $\mathcal{H}_x$  and if we want to stress its degree, say  $i$ , we write  $\partial_{i,x}$ . Let  $M$  be the normal closure of  $H$  in  $\Gamma$ . Thus  $M$  is the subgroup of the direct product  $\prod_{x \in X} H_x$  that contains the elements with all but finitely many trivial coordinates.

We can take the tensor product over  $\mathbb{Z}$  of the deleted complexes  $\mathcal{H}_x^{del}$  (i.e. in  $\mathcal{H}_x$  delete  $\mathbb{Z}$ ) for  $x \in X$  and obtain the deleted complex  $\mathcal{F}^{del}$ . This makes sense for infinite  $X$  as the direct limit of the tensor products of any finite number of the complexes  $\mathcal{H}_x$ . Note that we have fixed a linear order  $\leq$  on  $X$  and the tensor product  $\mathcal{H}_{x_1}^{del} \otimes \cdots \otimes \mathcal{H}_{x_t}^{del}$  is made for  $x_1 < \cdots < x_t$  and  $t \geq 1$ . By the Künneth formula we get a free resolution of the trivial  $\mathbb{Z}M$ -module  $\mathbb{Z}$

$$\mathcal{F} : \cdots \rightarrow \mathbb{Z}M \otimes W_i \xrightarrow{d_i} \mathbb{Z}M \otimes W_{i-1} \rightarrow \cdots \rightarrow \mathbb{Z}M \otimes W_1 \xrightarrow{d_1} \mathbb{Z}M \rightarrow \mathbb{Z} \rightarrow 0,$$

where  $W_i$  is a free abelian group with a basis  $Z_i$ ,  $Z_i$  is the disjoint union

$$Z_i = \bigcup (Y_1^{i_1}) \times (Y_2^{i_2}) \times \cdots \times (Y_s^{i_s}) \times X^{(i_1, i_2, \dots, i_s)},$$

over all possible  $s \geq 1$  such that  $i_1 + 2i_2 + \cdots + si_s = i$ ,  $i_j \geq 0$  and  $X^{(i_1, i_2, \dots, i_s)}$  is the set of  $s$ -tuples  $(X_1, \dots, X_s)$  of pairwise disjoint subsets  $X_1, \dots, X_s$  of  $X$  such that  $X_j$  has cardinality  $i_j$  for all  $1 \leq j \leq s$ . We write the elements of  $Z_i$  as formal products  $y_1 \cdots y_j x_1 \cdots x_j$ , which indicates that in the tensor product  $y_i \in Y = \bigcup_{i \geq 1} Y_i$  is taken from the complex  $\mathcal{H}_{x_i}$  and  $x_1 < \cdots < x_j$  are elements of  $X$ , recall we have fixed some linear order  $\leq$  on  $X$ . Indeed the element  $y_1 \cdots y_j x_1 \cdots x_j$  corresponds to the element  $B_1 \times \cdots \times B_s \times (X_1, \dots, X_s)$  of  $Z_i$ , where  $X_r$  is the set of those elements  $x_t$  of  $\{x_1, \dots, x_j\}$  for which  $y_t \in Y_r$ , we write  $X_r = \{x_{j_1}, \dots, x_{j_{i_r}}\}$  where  $x_{j_1} < \cdots < x_{j_{i_r}}$  and set  $B_r = (y_{j_1}, \dots, y_{j_{i_r}}) \in Y_r^{i_r}$ .

Sometimes it will be convenient to use general products  $y_1 \cdots y_j x_1 \cdots x_j$ , where  $x_1, \dots, x_j$  are pairwise different elements of  $X$  but we do not assume that  $x_1 < \cdots < x_j$ . We follow (11) and define for all  $1 \leq i \leq j-1$

$$y_1 \cdots y_j x_1 \cdots x_j = (-1)^{\deg(y_i) \deg(y_{i+1})} y_1 \cdots y_{i+1} y_i \cdots y_j x_1 \cdots x_{i+1} x_i \cdots x_j.$$

This completes the definition of the general products  $y_1 \cdots y_j x_1 \cdots x_j$  and shows that they belong to  $Z_i \cup -Z_i$ .

As  $X$  is a (left)  $G$ -set we get an action of  $G$  on the general products given by

$$(12) \quad g(y_1 \cdots y_j x_1 \cdots x_j) = y_1 \cdots y_j(gx_1) \cdots (gx_j).$$

Thus  $W_i$  is a semi-induced  $\mathbb{Z}G$ -module i.e.  $W_i = \mathbb{Z}_i Z_i$ . Furthermore since  $Y_i$  is a finite set for  $i \leq m$  and by Lemma 9 we deduce that  $W_i$  is finitely generated as  $\mathbb{Z}G$ -module. Using the other notation for the elements of  $Z_i$  we have for  $B_1 \times \cdots \times B_s \times (X_1, \dots, X_s) \in Z_i$  and  $g \in G$  that

$$g(B_1 \times \cdots \times B_s \times (X_1, \dots, X_s)) = \pm(gB_1 \times \cdots \times gB_s \times (gX_1, \dots, gX_s))$$

where  $gB_k$  is obtained from  $B_k \in Y_k^{i_k}$  by some permutation (possibly the trivial one) of the  $i_k$  coordinates. By Lemma 8(b)  $W_i$  is of type  $FP_{m-i}$  as  $\mathbb{Z}G$ -module if the stabilizer in  $G$  of the element  $B_1 \times \cdots \times B_s \times (X_1, \dots, X_s) \in Z_i$  is of type  $FP_{m-i}$  for all  $B_1 \times \cdots \times B_s \times (X_1, \dots, X_s) \in Z_i$ . Any such stabilizer is commensurable with the stabilizer in  $G$  of  $(X_1, \dots, X_s) \in X^{(i_1, i_2, \dots, i_s)}$  and by the proof of Lemma 9 the last stabilizer is of type  $FP_{m-i}$ . Note we have proved that  $W_i$  is of type  $FP_{m-i}$  as  $\mathbb{Z}G$ -module for  $1 \leq i \leq m$ . Then  $\mathbb{Z}\Gamma \otimes_{\mathbb{Z}G} W_i \simeq \mathbb{Z}M \otimes W_i$  is of type  $FP_{m-i}$  as  $\mathbb{Z}\Gamma$ -module.

Since the action of the symmetric group in (11) commutes with the differential (10) for a general product  $y_1 \cdots y_j x_1 \cdots x_j$  the differential of  $\mathcal{F}$  is

$$d_s(y_1 \cdots y_j x_1 \cdots x_j) = \left( \sum_{\deg(y_i) \geq 2} (-1)^{\deg(y_1) + \cdots + \deg(y_i)} y_1 \cdots \partial_{x_i}(y_i) \cdots y_j x_1 \cdots x_i \cdots x_j \right) +$$

$$(13) \quad \left( \sum_{\deg(y_i)=1} (-1)^{\deg(y_1) + \cdots + \deg(y_i)} y_1 \cdots \partial_{x_i}(y_i) \cdots y_j x_1 \cdots \widehat{x_i} \cdots x_j \right)$$

where  $s = \sum_{1 \leq i \leq j} \deg(y_i) \geq 1$ . Note that by (12) the above differential commutes with the  $G$ -action, hence commutes with the  $\Gamma$ -action. Then by Lemma 2 applied to the complex (obtained from  $\mathcal{F}$ )

$$(14) \quad \tilde{\mathcal{F}} : \cdots \rightarrow \mathbb{Z}M \otimes W_i \xrightarrow{d_i} \mathbb{Z}M \otimes W_{i-1} \rightarrow \cdots \rightarrow \mathbb{Z}M \otimes W_1 \xrightarrow{d_1} \text{Aug}(\mathbb{Z}M) \rightarrow 0,$$

we deduce that  $\text{Aug}(\mathbb{Z}M)$  is of type  $FP_{m-1}$  as  $\mathbb{Z}\Gamma$ -module, so by Lemma 6  $\Gamma$  is of type  $FP_m$ .  $\square$

**Lemma 11.** Let  $\Gamma = H \wr_X G$  be a group with  $H \neq 1, X \neq \emptyset$ . Then  $\Gamma$  is of type  $FP_2$  if and only if  $H$  is of type  $FP_2$ ,  $G$  is of type  $FP_2$  and  $G$  acts on  $X^i$  with stabilizers of type  $FP_{2-i}$  and with finitely many orbits for  $1 \leq i \leq 2$ .

*Proof.* By the previous theorem it remains to show that if  $\Gamma$  is  $FP_2$  then all stated conditions hold. This is a particular case of [4, Prop. 4.9] when  $X$  is  $G$ -transitive, the general case follows by the same argument.  $\square$

**Proposition 5.** Let  $m \geq 1$ ,  $\Gamma = H \wr_X G$  be a group of type  $FP_m$  such that  $H \neq 1, X \neq \emptyset$ ,  $H$  is of type  $FP_{m-1}$  and  $G$  acts on  $X^i$  with stabilizers of type  $FP_{m-i}$  and with finitely many orbits for  $1 \leq i \leq m$ . Then  $H$  is of type  $FP_m$ .

*Proof.* As pointed in [15] the case  $m = 1$  holds, The case  $m = 2$  is a particular case of Lemma 11. So we may assume that  $m \geq 3$ .

Let  $Z_i$  and  $Y_i$  be as in the proof of Proposition 4. Since  $H$  is of type  $FP_{m-1}$  we can assume that

$$Y_1, Y_2, \dots, Y_{m-1} \text{ are all finite}$$

and will prove that  $Y_m$  can be chosen finite. The description of  $Z_m$  and  $X^{(0, \dots, 0, 1)} = X$  imply that

$$\mathbb{Z}Z_m = T_{1,m} \oplus T_{2,m},$$

where  $T_{1,m} = \mathbb{Z}[Y_m \times X]$  and  $T_{2,m} = \mathbb{Z}[Z_m \setminus (Y_m \times X)]$  i.e.  $T_{1,m} = \mathbb{Z}Y_m \otimes \mathbb{Z}X$ . Observe that since  $Y_1, \dots, Y_{m-1}$  are all finite and  $G$  acts on  $X^i$  with finitely many orbits for  $1 \leq i \leq m$  we have that  $T_{2,m}$  is finitely generated as  $\mathbb{Z}G$ -module. Note that by the proof of Proposition 4  $\mathbb{Z}M \otimes \mathbb{Z}Z_i$  is  $FP_{m-i}$  as  $\mathbb{Z}\Gamma$ -module for  $1 \leq i \leq m-1$ . Since  $\Gamma$  is of type  $FP_m$  by Lemma 6 and Lemma 3 applied to the complex (14)  $\text{im}(d_m)$  is finitely generated as  $\mathbb{Z}\Gamma$ -module. Consider the splitting of the domain  $\mathbb{Z}M \otimes \mathbb{Z}Z_m$  of the differential  $d_m$  as

$$\mathbb{Z}M \otimes \mathbb{Z}Z_m = W_{1,m} \oplus W_{2,m},$$

where  $W_{1,m} = \mathbb{Z}M \otimes T_{1,m}$  and  $W_{2,m} = \mathbb{Z}M \otimes T_{2,m}$ . Since  $W_{2,m}$  is finitely generated as  $\mathbb{Z}\Gamma$ -module the condition that  $\text{im}(d_m)$  is finitely generated is equivalent with  $d_m(W_{1,m})/d_m(W_{1,m}) \cap d_m(W_{2,m})$  is finitely generated as  $\mathbb{Z}\Gamma$ -module.

Let  $w_1 \in W_{1,m}, w_2 \in W_{2,m}$  be such that  $d_m(w_1) = d_m(w_2)$ . Then  $w_1 - w_2 \in \ker(d_m) = \text{im}(d_{m+1})$ . Let

$$p : W_{1,m} \oplus W_{2,m} \rightarrow W_{1,m}$$

be the canonical projection. By the definition of  $W_{1,m}$  and (5.2) applied with  $s = m+1$  we get that

$$w_1 = p(w_1 - w_2) \in p(\text{im}(d_{m+1})) =$$

$$pd_{m+1}((\mathbb{Z}M \otimes \mathbb{Z}[Y_1 \times Y_m \times X^{(1,0,\dots,0,1)}]) \oplus (\mathbb{Z}M \otimes \mathbb{Z}[Y_{m+1} \times X])).$$

Then since  $d_{m+1}(\mathbb{Z}M \otimes \mathbb{Z}[Y_{m+1} \times X]) \subseteq W_{1,m}$  we have  $pd_{m+1}(\mathbb{Z}M \otimes \mathbb{Z}[Y_{m+1} \times X]) = d_{m+1}(\mathbb{Z}M \otimes \mathbb{Z}[Y_{m+1} \times X])$  and so

$$\begin{aligned} d_m(w_1) &\in d_m pd_{m+1}(\mathbb{Z}M \otimes \mathbb{Z}[Y_1 \times Y_m \times X^{(1,0,\dots,0,1)}]) + d_m pd_{m+1}(\mathbb{Z}M \otimes \mathbb{Z}[Y_{m+1} \times X]) = \\ &= d_m pd_{m+1}(\mathbb{Z}M \otimes \mathbb{Z}[Y_1 \times Y_m \times X^{(1,0,\dots,0,1)}]) + d_m d_{m+1}(\mathbb{Z}M \otimes \mathbb{Z}[Y_{m+1} \times X]) = \\ (15) \quad &= d_m pd_{m+1}(\mathbb{Z}M \otimes \mathbb{Z}[Y_1 \times Y_m \times X^{(1,0,\dots,0,1)}]). \end{aligned}$$

Recall that  $\partial_{j,x}$  denotes the differential of  $\mathcal{H}_x$  in dimension  $j$  and when we do not want to stress the dimension we omit the index  $j$  and use  $\partial_x$ . Note that for  $y_1 \in Y_1, y_m \in Y_m, x_1, x_m \in X, x_1 \neq x_m$

$$\begin{aligned} d_m pd_{m+1}(y_1 y_m x_1 x_m) &= d_m p(-\partial_{x_1}(y_1) y_m x_m + (-1)^{m+1} y_1 \partial_{x_m}(y_m) x_1 x_m) = \\ (16) \quad &= d_m(-\partial_{x_1}(y_1) y_m x_m) = (-1)^{m+1} \partial_{x_1}(y_1) \partial_{x_m}(y_m) x_m. \end{aligned}$$

Decompose  $M = S_x \times H_x$ , where  $S_x \subseteq \prod_{t \neq x} H_t$ . Thus in (16) we have  $\partial_{x_m}(y_m) \in \text{im}(\partial_{m,x_m}), \partial_{x_1}(y_1) \in \text{Aug}(\mathbb{Z}S_{x_m})$ . Then by (15) and (16)

$$\begin{aligned} d_m(w_1) &\in \bigoplus_{y_m \in Y_m, y_1 \in Y_1, x_1, x_m \in X, x_1 \neq x_m} \mathbb{Z}M \partial_{x_1}(y_1) \partial_{x_m}(y_m) x_m = \\ (17) \quad &= \bigoplus_{x_m \in X} \text{Aug}(\mathbb{Z}S_{x_m}) \mathbb{Z}H_{x_m} \text{im}(\partial_{m,x_m}) x_m = \bigoplus_{x \in X} \text{Aug}(\mathbb{Z}S_x) \text{im}(\partial_{m,x}) x =: J \end{aligned}$$

and

$$(18) \quad d_m(W_{1,m}) = \bigoplus_{x \in X} \mathbb{Z}M \operatorname{im}(\partial_{m,x})x = \bigoplus_{x \in X} (\mathbb{Z}S_x \mathbb{Z}H_x) \operatorname{im}(\partial_{m,x})x = \bigoplus_{x \in X} \mathbb{Z}S_x \operatorname{im}(\partial_{m,x})x.$$

Furthermore for every  $j \in J$  there is  $w_1 \in W_{1,m}$  such that there is some  $w_2 \in W_{2,m}$  with  $d_m(w_1) = d_m(w_2)$  and  $d_m(w_1) = j$ . Thus by (17) and (18)

$$K := d_m(W_{1,m})/J \simeq \bigoplus_{x \in X} \operatorname{im}(\partial_{m,x})x$$

Thus the condition that  $\operatorname{im}(d_m)$  is finitely generated as  $\mathbb{Z}\Gamma$ -module is equivalent with  $K$  is finitely generated as  $\mathbb{Z}\Gamma$ -module. Take a finite set  $D$  of generators of  $K$  such that  $D \subseteq \bigcup_{x \in X} \operatorname{im}(\partial_{m,x})x$ . Write  $d \in D$  as  $a_x x$  and define  $A_x \subseteq \operatorname{im}(\partial_{m,x})$  as the set of all possible  $a_x$ . Thus the set of the images of the finite set  $\bigcup_{x \in X} A_x$  in  $\operatorname{im}(\partial_m)$ , where we send canonically every  $H_x$  to  $H$  by just forgetting the index  $x$ , generates  $\operatorname{im}(\partial_m)$  as  $\mathbb{Z}H$ -module. Thus  $\operatorname{im}(\partial_m)$  is finitely generated as  $\mathbb{Z}H$ -module, so  $Y_m$  can be chosen finite.  $\square$

**Theorem 2.** Let  $m \geq 1$  and  $\Gamma = H \wr_X G$  be of type  $FP_m$ , where  $X \neq \emptyset$  and  $H$  has infinite abelianization. Then  $H$  is of type  $FP_m$  and  $G$  acts on  $X^i$  with stabilizers of type  $FP_{m-i}$  and with finitely many orbits for all  $1 \leq i \leq m$ .

*Proof.* We induct on  $m$ . The case  $m = 1$  is trivial as the property  $FP_1$  is equivalent with finite generation and then [15, Prop. 2.1] applies.

Since  $H$  has infinite abelianization  $\mathbb{Z}$  is a retract of  $H$ , so  $\Gamma_0 = \mathbb{Z} \wr_X G$  is a retract of  $\Gamma$ . By Lemma 5 since  $\Gamma$  is of type  $FP_m$  the group  $\Gamma_0$  is  $FP_m$  too. Then by Proposition 1 and Theorem 1  $G$  acts on  $X^i$  with finitely many orbits and stabilizers of type  $FP_{m-i}$  for all  $1 \leq i \leq m$ . Then Proposition 5 completes the inductive step.  $\square$

**Theorem 3.** Let  $\Gamma = H \wr_X G$  be a wreath product, where  $X \neq \emptyset$  and  $H$  has infinite abelianization. Then the following are equivalent :

1.  $\Gamma$  is of type  $FP_m$ ;
2.  $H$  is of type  $FP_m$ ,  $G$  is of type  $FP_m$ ,  $G$  acts on  $X^i$  with stabilizers of type  $FP_{m-i}$  and with finitely many orbits for all  $1 \leq i \leq m$ .

*Proof.* Follows from Lemma 5, Proposition 4 and Theorem 2.  $\square$

The following is a homotopy version of Theorem 3.

**Theorem 4.** Let  $\Gamma = H \wr_X G$  be a wreath product, where  $X \neq \emptyset$  and  $H$  has infinite abelianization. Then the following are equivalent :

1.  $\Gamma$  is of type  $F_m$ ;
2.  $H$  is of type  $F_m$ ,  $G$  is of type  $F_m$ ,  $G$  acts on  $X^i$  with stabilizers of type  $FP_{m-i}$  and with finitely many orbits for all  $1 \leq i \leq m$ .

*Remark.* Observe that in the condition 2 we have a mixture of both properties  $F_m$  and  $FP_m$  and not just the homotopical property  $F_m$ .

*Proof.* In the case  $m \leq 2$  this is the main result of [15]. So we can assume that  $m > 2$ . In general a group is of type  $FP_m$  if and only if it is finitely presented (i.e. is  $F_2$ ) and is of type  $FP_m$ . Then the result follows from Theorem 3 and the fact that the case  $m = 2$  of Theorem 4 was already proven in [15].  $\square$

## 6. AN EXAMPLE

Let  $F$  be the Richard Thompson group with infinite presentation

$$\langle x_0, x_1, x_2 \dots | x_j^{x_i} = x_{j+1} \text{ for } 0 \leq i < j \rangle.$$

Consider its realization as piecewise linear transformations of the interval  $[0, 1]$ , see [12]. Let  $X = (0, 1) \cap \mathbb{Z}[\frac{1}{2}]$ . Then  $F$  acts with finitely many orbits on  $X^i$  for every  $i \geq 1$  i.e. the  $F$  orbit of  $x = (x_1, \dots, x_i) \in X^i$  contains  $y = (y_1, \dots, y_i)$  if and only if there is a permutation  $\sigma \in S_i$  such that  $x_{\sigma(1)} \leq x_{\sigma(2)} \leq \dots \leq x_{\sigma(i)}$  and  $y_{\sigma(1)} \leq y_{\sigma(2)} \leq \dots \leq y_{\sigma(i)}$  and  $x_{\sigma(k)} = x_{\sigma(k+1)}$  if and only if  $y_{\sigma(k)} = y_{\sigma(k+1)}$ . Furthermore the stabilizer in  $F$  of the point  $x = (x_1, \dots, x_i) \in X^i$  is  $F^{j+1}$ , where  $j$  is the number of different coordinates of  $x$ . By [14]  $F$  is of type  $FP_\infty$ , so all stabilizers of the action of  $F$  on  $X^i$  are of type  $FP_\infty$ . The following result is a direct corollary of Theorem 3 and Theorem 4.

**Corollary 2.** Let  $X$  and  $F$  be as above and  $\Gamma = H \wr_X F$  be a wreath product. Then  $H$  has type  $FP_m$  (resp.  $F_m$ ) if and only if  $\Gamma$  is of type  $FP_m$  (resp.  $F_m$ ).

*Proof.* Follows directly from Proposition 4 and Proposition 5.  $\square$

## 7. BREDON TYPE $FP_m$ FOR WREATH PRODUCTS

**7.1. Preliminaries on Bredon homology.** In this section we study the homological finiteness Bredon type  $FP_m$ . Its homotopical  $\infty$ -version, the homotopical finiteness Bredon property  $\underline{E}_\infty$  was considered in [18]. Here we need the following homological version of Lück's result from [18].

**Theorem 5.** [17] A group  $\Gamma$  is of type  $FP_m$  if and only if  $\Gamma$  has finitely many conjugacy classes of finite subgroups and for every finite subgroup  $K$  of  $\Gamma$  the centralizer  $C_\Gamma(K)$  is of type  $FP_m$ .

### 7.2. Bredon type $FP_m$ for wreath products : a proof of Theorem C.

**Theorem 6.** Let  $\Gamma = H \wr_X G$  be a wreath product such that  $H$  is torsion-free. Then every finite subgroup of  $\Gamma$  is conjugate to a subgroup of  $G$ .

*Proof.* In this proof we consider  $X$  as a right  $G$ -set (if  $X$  is equipped with a left  $G$ -action it becomes a right  $G$ -action by  $xg = g^{-1}x$ ). Denote by  $\pi : \Gamma \rightarrow G$  the natural homomorphism. Let  $M$  be the normal closure of  $H$  in  $\Gamma$ . Elements  $\gamma$  of  $\Gamma$  may uniquely be written as  $\gamma = mg$  for  $m \in M$  and  $g \in G$ ; we have  $\pi(\gamma) = g$ , and write  $\gamma @ x := m_x$ , where  $m_x$  is the  $x$ -coordinate of  $m \in M \subseteq H^X$ . The assertion of the theorem is that for every finite subgroup  $K \leq \Gamma$  there exists  $\delta \in \Gamma$  such that  $(k^\delta) @ x = 1$  for all  $x \in X$ ,  $k \in K$ .

Consider first two points  $x, y \in X$  in the same orbit under  $\pi(K)$ . Then there exists a unique  $h_{x,y} \in H$  such that  $k @ x = h_{x,y}$  for all  $k \in K$  with  $x^{\pi(k)} = y$ . Indeed, consider two such  $k, k' \in K$  i.e.  $x^{\pi(k)} = y = x^{\pi(k')}$ . Then  $\pi(k'k^{-1})$  fixes  $x$ , and has finite order  $s$  because  $k'k^{-1}$  belongs to  $K$ ; so  $1 = (k'k^{-1})^s @ x = ((k'k^{-1}) @ x)^s$

implies  $1 = (k'k^{-1})@x = (k'@x)(k@x)^{-1}$ . Note the cocycle identity  $h_{x,y}h_{y,z} = h_{x,z}$ .

In each orbit  $\Omega = x^{\pi(K)}$  of  $\pi(K)$  on  $X$ , choose a representative  $x_\Omega$ , and define  $\delta \in H^X$  by  $\delta_x = h_{x,x_\Omega}$  whenever  $x$  lies in the orbit  $\Omega$ . Note that since  $K$  is finite there are only finitely many  $x, y \in X$  such that  $h_{x,y} \neq 1$ . Therefore  $\delta$  is a finitely supported function on  $X$  i.e.  $\delta \in M$ .

Now consider  $k \in K$ , and write  $y = x^{\pi(k)}$ . Note that  $x_\Omega = y_\Omega$ . Then

$$(k^\delta)@x = (\delta^{-1}@x)(k@x)(\delta@y) = h_{x,x_\Omega}^{-1} h_{x,y} h_{y,y_\Omega} = h_{x,x_\Omega}^{-1} h_{x,y_\Omega} = 1. \quad \square$$

**Theorem 7.** Let  $\Gamma = H \wr_X G$  be a wreath product,  $X \neq \emptyset$  and  $H$  be torsion-free, with infinite abelianization. Then  $\Gamma$  has type  $\underline{FP}_m$  if and only if the following conditions hold :

1.  $G$  has type  $\underline{FP}_m$ ;
2.  $H$  has type  $FP_m$ ;
3. for every finite subgroup  $K$  of  $G$  and every  $1 \leq i \leq m$  the centralizer  $C_G(K)$  acts on  $(K \backslash X)^i$  with stabilizers of type  $FP_{m-i}$  and with finitely many orbits.

*Proof.* By Theorem 5 we have to understand the centralizers and the conjugacy classes of finite subgroups in  $\Gamma$ . By Theorem 6 every finite subgroup  $K$  of  $\Gamma$  is conjugated to a finite subgroup  $K_0$  of  $G$ . Thus  $\Gamma$  has finitely many conjugacy classes of finite subgroups if and only if  $G$  has finitely many conjugacy classes of finite subgroups. This completes the proof for  $m = 0$ .

Furthermore for a finite subgroup  $K$  of  $G$  we have that

$$C_\Gamma(K) = C_M(K) \rtimes C_G(K) \simeq H \wr_{K \backslash X} C_G(K),$$

where  $M$  is the normal closure of  $H$  in  $\Gamma$  and we consider  $X$  as a left  $G$ -set. Then by Theorem 3 the centralizer  $C_\Gamma(K)$  is of type  $FP_m$  if and only if  $C_G(K)$  is  $FP_m$  and conditions 2 and 3 hold. Then Theorem 5 completes the proof.  $\square$

## 8. $\Sigma$ -INVARIANTS

**8.1. Some basic properties.** Following [6] we define an equivalence relation  $\sim$  on  $\text{Hom}(G, \mathbb{R}) \setminus \{0\}$  by  $\chi_1 \sim \chi_2$  if there is  $r \in \mathbb{R}_{>0}$  such that  $\chi_1 = r\chi_2$ . Denote by  $[\chi]$  the equivalence class of  $\chi$ . Then the character sphere  $S(G)$  is  $(\text{Hom}(G, \mathbb{R}) \setminus \{0\}) / \sim$ . By definition

$$\Sigma^m(G, \mathbb{Z}) = \{[\chi] \in S(G) \mid \mathbb{Z} \text{ is } FP_m \text{ as } \mathbb{Z}G_\chi - \text{module}\},$$

where  $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$ .

**Lemma 12.** Let  $m \geq 1$  be an integer,  $\Gamma = M \rtimes G$  be a finitely generated group and  $[\chi] \in \Sigma^m(\Gamma, \mathbb{Z})$  such that  $\chi(M) = 0$ . Then  $[\chi_0] \in \Sigma^m(G, \mathbb{Z})$ , where  $\chi_0$  is the restriction of  $\chi$  to  $G$ .

*Proof.* This is a monoid version of Lemma 5 and Aberg's idea ( i.e. the footnote from [1]) can be easily modified in this context. Indeed by [3] we have that  $[\chi] \in \Sigma^m(\Gamma, \mathbb{Z})$  is equivalent with

$$\text{Tor}_i^{\mathbb{Z}\Gamma}(\mathbb{Z}, \prod \mathbb{Z}\Gamma_\chi) = 0 \text{ for all } 1 \leq i \leq m-1$$

and the trivial  $\mathbb{Z}\Gamma_\chi$ -module  $\mathbb{Z}$  is finitely presented. Note that by the functoriality of  $\text{Tor}_i^{\mathbb{Z}S}(\mathbb{Z}, \prod \mathbb{Z}S)$  on  $S$  applied to the monoids  $S \in \{G_{\chi_0}, \Gamma_\chi\}$  and the fact that  $\Gamma_\chi = M \rtimes G_{\chi_0}$  we get that  $\text{Tor}_i^{\mathbb{Z}G_{\chi_0}}(\mathbb{Z}, \prod \mathbb{Z}G_{\chi_0}) = 0$  for all  $1 \leq i \leq m-1$ .

Furthermore the trivial  $\mathbb{Z}G_{\chi_0}$ -module  $\mathbb{Z}$  is obtained from the trivial  $\mathbb{Z}\Gamma_\chi$ -module  $\mathbb{Z}$  applying the right exact functor  $\mathbb{Z} \otimes_{\mathbb{Z}M}$ , hence the trivial  $\mathbb{Z}G_{\chi_0}$ -module  $\mathbb{Z}$  is finitely presented. Thus  $[\chi_0] \in \Sigma^m(G, \mathbb{Z})$ .  $\square$

**Lemma 13.** Let  $m \geq 1$  be an integer,  $\Gamma = M \rtimes G$  be a finitely generated group and  $\chi : \Gamma \rightarrow \mathbb{R}$  be a non-zero homomorphism such that  $\chi(M) = 0$ . Denote by  $\chi_0$  the restriction of  $\chi$  to  $G$ . Then  $[\chi] \in \Sigma^m(\Gamma, \mathbb{Z})$  if and only if  $[\chi_0] \in \Sigma^m(G, \mathbb{Z})$  and  $\text{Aug}(\mathbb{Z}M)$  is  $FP_{m-1}$  as  $\mathbb{Z}\Gamma_\chi$ -module.

*Proof.* In the proof of Lemma 6 substitute  $\Gamma$  with  $\Gamma_\chi$  and substitute  $G$  with  $G_{\chi_0}$ .  $\square$

The next lemma collects results from [21], that can be found in English version in [20].

**Lemma 14.** [21], [20] Let  $k \geq 0$  be an integer,  $\chi : G \rightarrow \mathbb{R}$  be a non-zero homomorphism,  $L$  a subgroup of  $G$ . Denote by  $\chi_0$  the restriction of  $\chi$  on  $L$ . Then

- (1)  $\mathbb{Z}G_\chi$  is a flat (right or left)  $\mathbb{Z}L_{\chi_0}$ -module and the inclusion map of  $\mathbb{Z}L_{\chi_0}$ -modules  $\mathbb{Z}L_{\chi_0} \rightarrow \mathbb{Z}G_\chi$  splits;
- (2) if  $\chi_0 \neq 0$  then for every  $\mathbb{Z}L$ -module  $V$  we have an isomorphism of left  $\mathbb{Z}G_\chi$ -modules  $\mathbb{Z}G \otimes_{\mathbb{Z}L} V \simeq \mathbb{Z}G_\chi \otimes_{\mathbb{Z}L_{\chi_0}} V$ ;
- (3) if  $\chi_0 \neq 0$  then for every  $\mathbb{Z}L$ -module  $V$  we have that  $\mathbb{Z}G \otimes_{\mathbb{Z}L} V$  is  $FP_m$  as  $\mathbb{Z}G_\chi$ -module if and only if  $V$  is  $FP_m$  as  $\mathbb{Z}L_{\chi_0}$ -module;
- (4) if  $L$  has finite index in  $G$  then for every  $\mathbb{Z}G$ -module  $V$  we have that  $V$  is  $FP_m$  as  $\mathbb{Z}G_\chi$ -module if and only if  $V$  is  $FP_m$  as  $\mathbb{Z}L_{\chi_0}$ -module.

*Proof.* Consider  $T$  a left transversal of  $L$  in  $G$  such that  $1 \in T$ . Then  $\mathbb{Z}G_\chi = \bigoplus_{t \in T} t\mathbb{Z}L_{\chi_0} \geq -\chi(t)$ , where  $L_{\chi_0 \geq r} = \{g \in L \mid \chi_0(g) \geq r\}$ . Since  $1 \in T$  the inclusion map of (right)  $\mathbb{Z}L_{\chi_0}$ -modules  $\mathbb{Z}L_{\chi_0} \rightarrow \mathbb{Z}G_\chi$  splits.

The rest of item 1. is [20, Lemma 9.1(i)]. Item 2 is [20, Lemma 9.1(ii)], item 3 is [20, Lemma 9.2] and item 4 is [20, Thm 9.3].  $\square$

## 8.2. Semi-induced modules.

**Lemma 15.** Let  $\chi : G \rightarrow \mathbb{R}$  be a non-zero homomorphism. Let  $S$  be a finitely generated semi-induced  $\mathbb{Z}G$ -module and consider the decomposition  $S \simeq \bigoplus_{i \in I} \mathbb{Z}G \otimes_{\mathbb{Z}K_i} V_i$  given by Lemma 8. Thus there is a subgroup  $G_i$  of  $K_i$  of index at most 2 that acts trivially on  $V_i$ . Then

- 1.  $S$  is finitely generated as  $\mathbb{Z}G_\chi$ -module if and only if for all  $i \in I$  for the restriction  $\chi_i$  of  $\chi$  on  $K_i$  we have  $\chi_i \neq 0$ ;
- 2.  $S$  is of type  $FP_m$  as  $\mathbb{Z}G_\chi$ -module if and only if for every  $i \in I$  for the restriction  $\tilde{\chi}_i$  of  $\chi$  to  $G_i$  we have that  $\tilde{\chi}_i \neq 0$  and the trivial  $\mathbb{Z}(G_i)_{\tilde{\chi}_i}$ -module  $\mathbb{Z}$  has type  $FP_m$  i.e.  $[\tilde{\chi}_i] \in \Sigma^m(G_i, \mathbb{Z})$ .

*Proof.* Note that item 1 is obvious. By item 1 we can assume that  $\tilde{\chi}_i \neq 0$ . By Lemma 14 (3)  $\mathbb{Z}G \otimes_{\mathbb{Z}K_i} V_i$  is  $FP_m$  as  $\mathbb{Z}G_\chi$ -module if and only if  $V_i$  is  $FP_m$  as  $\mathbb{Z}(K_i)_{\chi_i}$ -module.

It remains to consider only the case when  $K_i \neq G_i$ , so  $K_i/G_i$  is cyclic of order 2. Observe that  $V_i$  is the sign module with  $G_i$  acting trivially. By Lemma 14 (4)  $V_i$  is  $FP_m$  as  $\mathbb{Z}(K_i)_{\chi_i}$ -module if and only if  $V_i$  is  $FP_m$  as  $\mathbb{Z}(G_i)_{\tilde{\chi}_i}$ -module. The proof is completed by the fact that  $V_i$  is the trivial  $\mathbb{Z}G_i$ -module  $\mathbb{Z}$ .  $\square$

**Proposition 6.** Let  $M = \mathbb{Z}X$  be an induced  $\mathbb{Z}G$ -module and  $\chi : G \rightarrow \mathbb{R}$  a non-zero homomorphism. Then the following are equivalent :



1.  $\wedge^i M$  is of type  $FP_{m-i}$  as  $\mathbb{Z}G_\chi$ -module for all  $1 \leq i \leq m$ ;
2.  $\otimes^i M$  is of type  $FP_{m-i}$  as  $\mathbb{Z}G_\chi$ -module for all  $1 \leq i \leq m$ ;
3.  $\otimes^i M$  is finitely generated as  $\mathbb{Z}G_\chi$ -module via the diagonal action and for representatives  $\{G_1, \dots, G_{s_1}\}$  of all  $G$ -orbits of stabilizers of the diagonal action of  $G$  on  $X^i$ , the restriction  $\chi_j$  of  $\chi$  on  $G_j$  is non-zero and  $[\chi_j] \in \Sigma^{m-i}(G_j, \mathbb{Z})$  for all  $1 \leq i \leq m, 1 \leq j \leq s_1$ ;
4.  $\wedge^i M$  is finitely generated as  $\mathbb{Z}G_\chi$ -module via the diagonal action and for representatives  $\{\tilde{G}_1, \dots, \tilde{G}_{s_2}\}$  of all  $G$ -orbits of stabilizers of the action of  $G$  on the  $i$ -element subsets of  $X$ , the restriction  $\tilde{\chi}_j$  of  $\chi$  on  $\tilde{G}_j$  is non-zero and  $[\tilde{\chi}_j] \in \Sigma^{m-i}(\tilde{G}_j, \mathbb{Z})$  for all  $1 \leq i \leq m, 1 \leq j \leq s_2$ ;
5.  $G \backslash X^i$  is finite and for representatives  $\{G_1, \dots, G_{s_1}\}$  of all  $G$ -orbits of stabilizers of the diagonal action of  $G$  on  $X^i$  the restriction  $\chi_j$  of  $\chi$  on  $G_j$  is non-zero and  $[\chi_j] \in \Sigma^{m-i}(G_j, \mathbb{Z})$  for all  $1 \leq i \leq m, 1 \leq j \leq s_1$ .

*Proof.* We say that the monoid  $G_\chi$  acts with finitely many orbits on a set  $Y$  if there is a finite subset  $Y_0$  of  $Y$  such that  $G_\chi Y_0 = Y$ . Then to prove Proposition 6 it is sufficient to repeat the proof of Proposition 1 substituting  $G$  with  $G_\chi$ , and apply Lemma 14 and Lemma 15 instead of Lemma 8.  $\square$

**Theorem 8.** Let  $\Gamma = H \wr_X G$  be a wreath product of type  $FP_m$ ,  $X \neq \emptyset$  and  $H$  has infinite abelianization. Let  $\chi : \Gamma \rightarrow \mathbb{R}$  be a non-zero character such that  $\chi(H) = 0$ . Then the following are equivalent :

1.  $[\chi] \in \Sigma^m(\Gamma, \mathbb{Z})$ ;
2.  $[\chi|_G] \in \Sigma^m(G, \mathbb{Z})$  and for representatives  $\{G_1, \dots, G_{s_1}\}$  of all  $G$ -orbits of stabilizers of the diagonal action of  $G$  on  $X^i$  the restriction  $\chi_j$  of  $\chi$  on  $G_j$  is non-zero and  $[\chi_j] \in \Sigma^{m-i}(G_j, \mathbb{Z})$  for all  $1 \leq i \leq m, 1 \leq j \leq s_1$ .

*Proof.* The proof is the same as the proof of Theorem 3 substituting  $G$  with  $G_\chi$ .

The fact that 2. implies 1. is a monoid version of Proposition 4. Note that in the proof of Proposition 4 we use that some stabilizers are commensurable. Thus they have the same homological type  $FP_k$ . This step requires justification in its  $\Sigma$ -version, but this is the content of Lemma 14 (4).

The fact that item 1 implies item 2 follows from Lemma 12 applied to the retracts  $G$  and  $\Gamma_1 = \mathbb{Z} \wr_X G$  of  $\Gamma$ . Indeed we get that  $[\chi|_G] \in \Sigma^m(G, \mathbb{Z})$  and  $[\chi|_{\Gamma_1}] \in \Sigma^m(\Gamma_1, \mathbb{Z})$ . By the monoid version of Theorem 1 the result holds for  $\Gamma_1$ , so condition 5 from Proposition 6 holds.  $\square$

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